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On the superharmonic instability of surface gravity waves on fluid of finite depth

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The linear stability of two-dimensional surface gravity waves on fluid of finite depth is investigated for superharmonic disturbances. For this problem, Zufiria & Saffman (*Stud. Appl. Maths* vol. 74, 1986, p. 259) suggested that an exchange of stability occurs when the total wave energy becomes stationary as a function of wave speed for fixed 'Bernoulli constant'. In defining the potential energy of the above total wave energy, the surface displacement was measured from the quiescent surface with the same 'Bernoulli constant'. We have re-examined this problem both analytically and numerically, and found that the above 'Bernoulli constant' must be replaced by 'mean surface height' for the statement to be valid.

1. Introduction

We consider the linear stability of two-dimensional periodic surface gravity waves on fluid of finite depth under the restriction that disturbances are periodic in one wavelength of the basic wave, so called superharmonic stability. The previous studies on the superharmonic stability of surface gravity waves are mainly for deep fluid, or fluid of infinite depth. Longuet-Higgins (1978) was the first to treat this problem. Tanaka (1983) found numerically that an exchange of stability occurs when the total wave energy becomes stationary as a function of wave speed. Saffman (1985) gave its analytical proof using Zakharov's Hamiltonian formulation (Zakharov 1968). Detailed numerical results of the linear stability analysis were then presented by Longuet-Higgins & Tanaka (1997). Thus, great progress was achieved in understanding the superharmonic stability of surface gravity waves on deep fluid.

Waves on fluid of finite depth were treated first by Zufiria & Saffman (1986). They extended Saffman's theory, and obtained the analytical result that an exchange of stability occurs when the total wave energy becomes stationary as a function of wave speed for fixed 'Bernoulli constant'. This fixed parameter must be given, since the periodic surface waves on fluid of finite depth are characterized by two parameters, unlike those on deep fluid that are characterized by a single parameter. In addition, the surface displacement, which constitutes the above total wave energy, was measured from the quiescent surface with the same 'Bernoulli constant' in their analysis. On the other hand, according to our analysis, the result was fundamentally different. That is, the above-mentioned 'Bernoulli constant' must be replaced by 'mean surface height' for the above statement to be true.

In this paper, we explain our analysis and point out a problem of Zufiria & Saffman's analysis. Instead of introducing the Hamiltonian formulation with complex canonical variables, we here solve the eigenvalue problem directly using an asymptotic analysis for small eigenvalues, or small growth rates of disturbances (Kataoka 2006). This direct method has the advantage over the Hamiltonian formulation in that the physical meaning of each term is readily understandable. After describing our analysis, we verify the result numerically. Concluding remarks follow.

2. Basic equations

Let us consider the two-dimensional irrotational motion of an incompressible ideal fluid with a free surface. The fluid, which lies on a flat bottom, is subject to a uniform gravitational acceleration g. The effects of surface tension are neglected. All variables are non-dimensionalized using g and a reference length L. For the moment we leave L unspecified (it is specified after (2.7)). Introducing the two-dimensional Cartesian coordinates x, y with y vertically upward and their origin located at the bottom, we obtain the following set of dimensionless governing equations for the fluid motion:

$$\Delta\phi = 0 \quad (0 < y < \eta), \tag{2.1}$$

with boundary conditions

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y} \quad \text{at } y = \eta, \tag{2.2}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] + \eta = f(t) \quad \text{at } y = \eta, \tag{2.3}$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = 0, \tag{2.4}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2.5}$$

t is the time, $\phi(x, y, t)$ is the velocity potential, $\eta(x, t)$ is the height of the surface from the bottom, and $f(t)$ is a given function of t.

Let us consider the steady solution of (2.1)–(2.4) in the following form:

$$\phi = -cx + \Phi_s(x, y;\, b, c), \qquad \eta = \eta_s(x;\, b, c), \tag{2.6a, b}$$

with

$$f(t) = b + \tfrac{1}{2}c^2, \tag{2.7}$$

where b and c are given positive constants, and the functions Φ_s and η_s are periodic in x with unit period. Solution (2.6a, b) represents a periodic wave that propagates steadily against a uniform stream of constant velocity $-c$ in the x-direction, and whose Bernoulli constant (relative to the uniform stream $\phi = -cx$) is b. We consider the solution of this class with all crests being of the same height and all troughs of a different same height, and call it 'basic wave solution'. Since the non-dimensional wavelength of this wave is set at unity, the reference length L of the system mentioned at the first paragraph of this section is the wavelength of this basic wave. Substituting (2.6a, b) into (2.1)–(2.4), we obtain a set of governing equations for Φ_s and η_s as follows:

$$\frac{\partial^2 \Phi_s}{\partial x^2} + \frac{\partial^2 \Phi_s}{\partial y^2} = 0 \quad \text{for } 0 < y < \eta_s, \tag{2.8}$$

$$\left(-c+\frac{\partial \Phi_s}{\partial x}\right)\frac{\mathrm{d}\eta_s}{\mathrm{d}x}=\frac{\partial \Phi_s}{\partial y} \quad \text{at } y=\eta_s, \tag{2.9}$$

$$-c\frac{\partial \Phi_s}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial \Phi_s}{\partial x}\right)^2+\left(\frac{\partial \Phi_s}{\partial y}\right)^2\right]+\eta_s=b \quad \text{at } y=\eta_s, \tag{2.10}$$

$$\frac{\partial \Phi_s}{\partial y}=0 \quad \text{at } y=0. \tag{2.11}$$

Since Φ_s is periodic in x with unit period, we can impose the condition $\int_{-1/2}^{1/2}\Phi_s\,\mathrm{d}x=0$ at one particular level $y=$ constant beneath the wave troughs, and hence (from (2.8) and (2.11)) at all levels within the fluid. Then, the origin of the x-coordinate can be chosen such that Φ_s is odd and η_s is even in x, since this basic wave is symmetric about its crest and trough according to Garabedian (1965).

In order to investigate the linear stability of the above basic wave with respect to disturbances that are periodic in one wavelength of the basic wave, the solution of (2.1)–(2.4) is expressed as summation of the basic wave solution (2.6*a*, *b*) and its disturbances:

$$\phi=-cx+\Phi_s+\hat{\phi}(x,y)\exp(\lambda t), \qquad \eta=\eta_s+\hat{\eta}(x)\exp(\lambda t), \tag{2.12a, b}$$

where λ is a complex constant to be determined, and $\hat{\phi}$ and $\hat{\eta}$ are periodic in x with unit period. Substituting (2.12*a*, *b*) into (2.1)–(2.4) and linearizing with respect to $(\hat{\phi}, \hat{\eta})$, we obtain the following set of linear equations for $(\hat{\phi}, \hat{\eta})$:

$$\Delta\hat{\phi}=0 \quad \text{for } 0<y<\eta_s, \tag{2.13}$$

$$\mathrm{L}_1[\hat{\phi},\hat{\eta}]=-\lambda\hat{\eta} \quad \text{at } y=\eta_s, \tag{2.14}$$

$$\mathrm{L}_2[\hat{\phi},\hat{\eta}]=-\lambda\hat{\phi} \quad \text{at } y=\eta_s, \tag{2.15}$$

$$\frac{\partial\hat{\phi}}{\partial y}=0 \quad \text{at } y=0, \tag{2.16}$$

where L_1 and L_2 are linear operators defined by

$$\mathrm{L}_1[\hat{\phi},\hat{\eta}]=\left(-\frac{\partial}{\partial y}+\frac{\mathrm{d}\eta_s}{\mathrm{d}x}\frac{\partial}{\partial x}\right)\hat{\phi}+\left[\frac{\partial^2\Phi_s}{\partial x^2}+\frac{\partial^2\Phi_s}{\partial x\partial y}\frac{\mathrm{d}\eta_s}{\mathrm{d}x}+\left(-c+\frac{\partial \Phi_s}{\partial x}\right)\frac{\mathrm{d}}{\mathrm{d}x}\right]\hat{\eta}, \tag{2.17}$$

$$\mathrm{L}_2[\hat{\phi},\hat{\eta}]=\left[\left(-c+\frac{\partial \Phi_s}{\partial x}\right)\frac{\partial}{\partial x}+\frac{\partial \Phi_s}{\partial y}\frac{\partial}{\partial y}\right]\hat{\phi}+\left[\left(-c+\frac{\partial \Phi_s}{\partial x}\right)\frac{\partial^2\Phi_s}{\partial x\partial y}+\frac{\partial \Phi_s}{\partial y}\frac{\partial^2\Phi_s}{\partial y^2}+1\right]\hat{\eta}. \tag{2.18}$$

Equations (2.13)–(2.16) together with periodic conditions for $(\hat{\phi}, \hat{\eta})$ constitute an eigenvalue problem for $(\hat{\phi}, \hat{\eta})$ whose eigenvalue is λ. When this problem possesses a solution whose eigenvalue λ has positive real part, the corresponding basic wave is superharmonically unstable.

Before proceeding to the next section, we define two dimensionless integral properties of the basic wave: the mean surface height from the bottom and the total wave energy per wave cycle. The mean surface height $\bar{\eta}_s$ from the bottom is

$$\bar{\eta}_s(b,c)\equiv\int_{-1/2}^{1/2}\eta_s\,\mathrm{d}x, \tag{2.19}$$

which differs from b in general (see tables 1 and 2 below for specific examples). For the total wave energy, we must decide how the surface displacement is defined, and we will give two different definitions. The first one is defined in terms of surface displacement measured from a quiescent surface $y = b$ of fixed Bernoulli constant b. The corresponding total wave energy E per wave cycle is

$$E\,(b,c) \equiv \frac{1}{2}\int_{-1/2}^{1/2}\left\{\int_0^{\eta_s}\left[\left(\frac{\partial \Phi_s}{\partial x}\right)^2 + \left(\frac{\partial \Phi_s}{\partial y}\right)^2\right]\mathrm{d}y + (\eta_s - b)^2\right\}\mathrm{d}x. \tag{2.20}$$

Zufiria & Saffman (1986) used this definition. The second one is defined in terms of surface displacement measured from the mean surface height $y = \bar{\eta}_s$. The corresponding total wave energy $\bar{E}$ per wave cycle is

$$\begin{aligned}\bar{E}\,(\bar{\eta}_s,c) &\equiv \frac{1}{2}\int_{-1/2}^{1/2}\left\{\int_0^{\eta_s}\left[\left(\frac{\partial \Phi_s}{\partial x}\right)^2 + \left(\frac{\partial \Phi_s}{\partial y}\right)^2\right]\mathrm{d}y + (\eta_s - \bar{\eta}_s)^2\right\}\mathrm{d}x \\ &= E\,(b,c) - \frac{(\bar{\eta}_s - b)^2}{2}.\end{aligned} \tag{2.21}$$

We will see that the latter definition (2.21) is more significant in discussing the superharmonic instability.

3. Asymptotic analysis

We will investigate how an exchange of stability occurs, or how the eigenvalue λ in (2.13)–(2.16) goes through zero from pure imaginary to real. To this end, we make an asymptotic analysis of (2.13)–(2.16) for small $|\lambda|$. That is, we seek the asymptotic solution $(\hat{\phi}, \hat{\eta})$ as $|\lambda| \to 0$ of (2.13)–(2.16) with an appreciable variation in x and y of order unity ($\partial\hat{\phi}/\partial x = O(\hat{\phi})$, $\partial\hat{\phi}/\partial y = O(\hat{\phi})$, and $\mathrm{d}\hat{\eta}/\mathrm{d}x = O(\hat{\eta})$), in the following power series of λ:

$$\hat{\phi} = \hat{\phi}_0 + \lambda\hat{\phi}_1 + \lambda^2\hat{\phi}_2 + \cdots, \qquad \hat{\eta} = \hat{\eta}_0 + \lambda\hat{\eta}_1 + \lambda^2\hat{\eta}_2 + \cdots. \tag{3.1a, b}$$

Substituting the series (3.1a, b) into (2.13)–(2.16) and arranging the same-order terms in $|\lambda|$, we obtain a series of sets of equations for $(\hat{\phi}_n, \hat{\eta}_n)$ $(n = 0, 1, 2, \ldots)$:

$$\Delta\hat{\phi}_n = 0 \quad \text{for } 0 < y < \eta_s, \tag{3.2}$$

$$\mathrm{L}_1[\hat{\phi}_n, \hat{\eta}_n] = -\hat{\eta}_{n-1} \quad \text{at } y = \eta_s, \tag{3.3}$$

$$\mathrm{L}_2[\hat{\phi}_n, \hat{\eta}_n] = -\hat{\phi}_{n-1} \quad \text{at } y = \eta_s, \tag{3.4}$$

$$\frac{\partial\hat{\phi}_n}{\partial y} = 0 \quad \text{at } y = 0, \tag{3.5}$$

where Δ, L_1, and L_2 are defined by (2.5), (2.17), and (2.18) respectively. Note that $\hat{\phi}_n$ and $\hat{\eta}_n$ $(n = 0, 1, 2, \ldots)$ are periodic in x with unit period and $\hat{\phi}_{-1} = \hat{\eta}_{-1} = 0$.

At $n = 0$, the above set of equations (3.2)–(3.5) is homogeneous, and has the following solution:

$$\hat{\phi}_0 = \alpha + \beta\frac{\partial \Phi_s}{\partial x}, \quad \hat{\eta}_0 = \beta\frac{\mathrm{d}\eta_s}{\mathrm{d}x}, \tag{3.6}$$

where α and β are arbitrary constants to be determined. $(\hat{\phi}_0, \hat{\eta}_0) = (\partial\Phi_s/\partial x, \mathrm{d}\eta_s/\mathrm{d}x)$ comes from invariance of the system (2.8)–(2.11) under the horizontal shift.

At $n = 1, 2, \ldots$, the set of equations (3.2)–(3.5) is inhomogeneous. Its homogeneous part, which is the same as that at $n = 0$, has two non-trivial fundamental solutions $(\hat{\phi}_n, \hat{\eta}_n) = (1, 0)$ and $(\partial \Phi_s/\partial x,\ \mathrm{d}\eta_s/\mathrm{d}x)$. Therefore, for this set of inhomogeneous equations (3.2)–(3.5) at $n = 1, 2, \ldots$ to have a solution, its inhomogeneous terms $-\hat{\eta}_{n-1}$ and $-\hat{\phi}_{n-1}$ $(n = 1, 2, \ldots)$ on the right-hand sides of (3.3) and (3.4) must satisfy some relations. Since the homogeneous part satisfies

$$\int_{-1/2}^{1/2} \mathrm{d}x \int_0^{\eta_s} \Delta \hat{\phi}_n \mathrm{d}y + \int_{-1/2}^{1/2} [\mathrm{L}_1[\hat{\phi}_n, \hat{\eta}_n]]_{y=\eta_s} \mathrm{d}x = 0,$$

$$\int_{-1/2}^{1/2} \mathrm{d}x \int_0^{\eta_s} \frac{\partial \Phi_s}{\partial x} \Delta \hat{\phi}_n \mathrm{d}y + \int_{-1/2}^{1/2} \left[\frac{\partial \Phi_s}{\partial x} \mathrm{L}_1[\hat{\phi}_n, \hat{\eta}_n] - \frac{\mathrm{d}\eta_s}{\mathrm{d}x} \mathrm{L}_2[\hat{\phi}_n, \hat{\eta}_n] \right]_{y=\eta_s} \mathrm{d}x = 0,$$

its inhomogeneous terms $-\hat{\eta}_{n-1}$ and $-\hat{\phi}_{n-1}$ must satisfy the following relations (solvability conditions):

$$\sum_{m=1}^{n} \lambda^m \int_{-1/2}^{1/2} \hat{\eta}_{m-1} \mathrm{d}x = o(|\lambda|^n), \tag{3.7a}$$

$$\sum_{m=1}^{n} \lambda^m \int_{-1/2}^{1/2} \left[-\frac{\partial \Phi_s}{\partial x} \hat{\eta}_{m-1} + \frac{\mathrm{d}\eta_s}{\mathrm{d}x} \hat{\phi}_{m-1} \right]_{y=\eta_s} \mathrm{d}x = o(|\lambda|^n) \tag{3.7b}$$

$$(n = 1, 2, \ldots)$$

where the quantities in the square brackets with subscript $y = \eta_s$ are evaluated at $y = \eta_s$, and $o(|\lambda|^n)$ represents terms of order smaller than $|\lambda|^n$.

Substituting (3.6) into (3.7a, b) at $n = 1$, we find that the solvability conditions at $n = 1$ are identically satisfied, and the solution of (3.2)–(3.5) at $n = 1$ is

$$\hat{\phi}_1 = -\alpha \frac{\partial \Phi_s}{\partial b} - \beta \frac{\partial \Phi_s}{\partial c}, \qquad \hat{\eta}_1 = -\alpha \frac{\partial \eta_s}{\partial b} - \beta \frac{\partial \eta_s}{\partial c}, \tag{3.8a, b}$$

where $\partial/\partial b$ denotes the derivative with respect to b for fixed x, y, c, and $\partial/\partial c$ denotes that with respect to c for fixed x, y, and b. The homogeneous solution (3.6) multiplied by an arbitrary constant is omitted in (3.8a, b), since it can be included in the leading-order solution (3.6). One can check that the solution (3.8a, b) satisfies (3.2)–(3.5) at $n = 1$ by differentiating (2.8)–(2.11) with respect to b or c.

Substituting (3.8a, b) into (3.7a, b) at $n = 2$, we find that the solvability conditions at $n = 2$ are

$$\lambda^2 \left(\alpha \frac{\partial \bar{\eta}_s}{\partial b} + \beta \frac{\partial \bar{\eta}_s}{\partial c} \right) = o(|\lambda|^2), \tag{3.9a}$$

$$\frac{\lambda^2}{c} \left[\alpha \left(\frac{\partial E}{\partial b} + \bar{\eta}_s - b \right) + \beta \frac{\partial E}{\partial c} \right] = o(|\lambda|^2), \tag{3.9b}$$

where $\bar{\eta}_s$ and E are defined by (2.19) and (2.20), respectively. From (3.9a, b), the condition for the existence of solution $(\hat{\phi}_2, \hat{\eta}_2)$ with non-zero (α, β) is

$$\lambda^2 W_1 = o(|\lambda|^2), \tag{3.10}$$

where

$$W_1 = \frac{\partial E}{\partial c} - \left(\frac{\partial E}{\partial b} + \bar{\eta}_s - b \right) \left(\frac{\partial \bar{\eta}_s}{\partial b} \right)^{-1} \frac{\partial \bar{\eta}_s}{\partial c} \tag{3.11a}$$

$$= \frac{\partial E}{\partial c} - c\left(\frac{\partial \bar{\eta}_s}{\partial b}\right)^{-1}\left(\frac{\partial \bar{\eta}_s}{\partial c}\right)^2, \tag{3.11b}$$

and the following relation is used to derive (3.11b):

$$\frac{1}{c}\left(\frac{\partial E}{\partial b} + \bar{\eta}_s - b\right) = \frac{\partial \bar{\eta}_s}{\partial c}. \tag{3.12}$$

One can obtain (3.12) by calculating $\int_{-1/2}^{1/2}[-(\mathrm{d}\eta_s/\mathrm{d}x)\partial\Phi_s/\partial b + (\partial\Phi_s/\partial x)\partial\eta_s/\partial b]_{y=\eta_s}$ $\mathrm{d}x$ in two different ways. The first is to replace $\mathrm{d}\eta_s/\mathrm{d}x$ and $\partial\Phi_s/\partial x$ inside the above integral by the corresponding terms in (2.9) and (2.10), which gives the left-hand side of (3.12). The second way is to replace $\mathrm{d}\eta_s/\mathrm{d}x$ and $\partial\Phi_s/\partial x$ by the left-hand-side terms of (3.3) and (3.4) at $n = 1$ with (3.8a, b) at $\alpha = 0$ and $\beta = -1$, or $\mathrm{L}_1[\partial\Phi_s/\partial c, \partial\eta_s/\partial c]$ and $\mathrm{L}_2[\partial\Phi_s/\partial c, \partial\eta_s/\partial c]$, and integrate the result by parts, which gives the right-hand side of (3.12). When condition (3.10) is satisfied, the solution $(\hat{\phi}_2, \hat{\eta}_2)$ of (3.2)–(3.5) at $n = 2$ exists, although its explicit form cannot be obtained. Here we only find that $\hat{\phi}_2$ is even and $\hat{\eta}_2$ is odd in x by examining the order of the differential operators with respect to x of (3.2)–(3.5) and noting the parity in x of the basic wave solution (Φ_s, η_s) (Φ_s is odd and η_s is even; see the statement after (2.11)) and that of the inhomogeneous terms $(-\hat{\phi}_1, -\hat{\eta}_1)$ ($\hat{\phi}_1$ is odd and $\hat{\eta}_1$ is even).

Let us proceed to the next order $n = 3$. From the parity in x of (Φ_s, η_s) and $(\hat{\phi}_2, \hat{\eta}_2)$ mentioned above, the integrands of $O(|\lambda|^3)$ in the solvability conditions (3.7a, b) are odd in x, so that the corresponding integrals vanish. The solvability conditions at $n = 3$, therefore, remain the same form as those (3.7a, b) at $n = 2$, and the solution $(\hat{\phi}_3, \hat{\eta}_3)$ of (3.2)–(3.5) at $n = 3$ exists. By examining the order of differential operators with respect to x of (3.2)–(3.5) and noting the parity in x of the basic wave solution (Φ_s, η_s) (Φ_s is odd and η_s is even) and that of the inhomogeneous terms $(-\hat{\phi}_2, -\hat{\eta}_2)$ ($\hat{\phi}_2$ is even and $\hat{\eta}_2$ is odd), we find that $\hat{\phi}_3$ is odd and $\hat{\eta}_3$ is even in x.

Let us proceed to the next order $n = 4$. The solvability conditions at $n = 4$ are written in the following form:

$$\lambda^2\left(\alpha\frac{\partial \bar{\eta}_s}{\partial b} + \beta\frac{\partial \bar{\eta}_s}{\partial c}\right) + \lambda^4(\alpha S_1 + \beta S_2) = o(|\lambda|^4), \tag{3.13a}$$

$$\frac{\lambda^2}{c}\left[\alpha\left(\frac{\partial E}{\partial b} + \bar{\eta}_s - b\right) + \beta\frac{\partial E}{\partial c}\right] + \lambda^4(\alpha S_3 + \beta S_4) = o(|\lambda|^4), \tag{3.13b}$$

where S_1, S_2, S_3, and S_4 are obtained by equating terms with α and those with β of the following equations:

$$\alpha S_1 + \beta S_2 = \int_{-1/2}^{1/2} -\hat{\eta}_3\,\mathrm{d}x, \tag{3.14a}$$

$$\alpha S_3 + \beta S_4 = \int_{-1/2}^{1/2}\left[-\frac{\partial \Phi_s}{\partial x}\hat{\eta}_3 + \frac{\mathrm{d}\eta_s}{\mathrm{d}x}\hat{\phi}_3\right]_{y=\eta_s}\mathrm{d}x. \tag{3.14b}$$

Condition (3.13b) with $\alpha = 0$ is that obtained by Zufiria & Saffman (1986), and this condition indicates that an exchange of stability occurs at $\partial E/\partial c = 0$. However, another condition (3.13a) must be taken into account, since it appears as the solvability condition to be satisfied. Zufiria & Saffman (1986) overlooked another eigenfunction $(\mathrm{i}\sqrt{|k|}\delta(k), \mathrm{i}\sqrt{|k|}\delta(k))$ of the adjoint operator L^+ with $\sigma = 0$ in their analysis (see (24) of their paper) where $\delta(k)$ is the Dirac delta function. This eigenfunction leads to the solvability condition corresponding to (3.13a). Physically, this equation (3.13a) has the role of balancing the mass of fluid when the mean surface height $\bar{\eta}_s\,(b, c)$ of the basic wave changes due to its modulation.

From (3.13a, b), the condition for the existence of solution $(\hat{\phi}_4, \hat{\eta}_4)$ with non-zero (α, β) is

$$\lambda^2 W_1 + \lambda^4 W_2 = o(|\lambda|^4), \tag{3.15}$$

where W_1 is given by (3.11a, b), and W_2 is expressed as

$$W_2 = c\left[\left(\frac{\partial\bar{\eta}_s}{\partial c}\right)^2\left(\frac{\partial\bar{\eta}_s}{\partial b}\right)^{-2} S_1 - \frac{\partial\bar{\eta}_s}{\partial c}\left(\frac{\partial\bar{\eta}_s}{\partial b}\right)^{-1}(S_2 + S_3) + S_4\right], \tag{3.16}$$

with the aid of (3.12). Since $\hat{\phi}_3$ and Φ_s are odd, and $\hat{\eta}_3$ and η_s are even in x, the integrands on the right-hand sides of (3.14a, b) are even in x, so that the corresponding integrals rarely vanish. Therefore, W_2 defined by (3.16) is non-zero in general, and (3.15) gives

$$\lambda = \begin{cases} 0, \ \pm\sqrt{\left|\dfrac{W_1}{W_2}\right|} & \text{when } \dfrac{W_1}{W_2} < 0, \\ 0, \ \pm \mathrm{i}\sqrt{\dfrac{W_1}{W_2}} & \text{when } \dfrac{W_1}{W_2} > 0. \end{cases} \tag{3.17}$$

The solution (3.17), which is valid for $|\lambda| \ll 1$, indicates that an exchange of stability occurs at $W_1 = 0$ where W_1 is defined by (3.11a, b). The critical amplitude is different from the point where $\partial E/\partial c = 0$, since the second terms on the right-hand sides of (3.11a, b) are non-zero. Thus, an exchange of stability occurs not at $\partial E/\partial c = 0$ but at $W_1 = 0$. If we evaluate W_1 in terms of another definition (2.21) of the total wave energy $\bar{E}\,(\bar{\eta}_s, c)$ of the basic wave, we find

$$W_1 = \left.\frac{\partial \bar{E}}{\partial c}\right|_{\bar{\eta}_s}, \tag{3.18}$$

where $\partial/\partial c|_{\bar{\eta}_s}$ denotes the derivative with respect to c for fixed $\bar{\eta}_s$. Thus, an exchange of stability occurs at the stationary value of the total wave energy $\bar{E}$ for fixed $\bar{\eta}_s$, where $\bar{E}$ is defined by (2.21) and $\bar{\eta}_s$ is the mean surface height defined by (2.19). This is the main result of this paper.

If we introduce the impulse I of the basic wave defined by

$$I\,(\bar{\eta}_s, c) = \int_{-1/2}^{1/2} \mathrm{d}x \int_0^{\eta_s} \frac{\partial \Phi_s}{\partial x}\mathrm{d}y, \tag{3.19}$$

W_1 is expressed as

$$W_1 = c\left.\frac{\partial I}{\partial c}\right|_{\bar{\eta}_s}, \tag{3.20}$$

which means that an exchange of stability occurs at the stationary value of I for fixed $\bar{\eta}_s$, or $\partial I/\partial c|_{\bar{\eta}_s} = 0$.

4. Numerical verification

We will verify the above result numerically. The numerical method is fundamentally based on Tanaka (1986), which was originally devised for investigating the stability of solitary waves. We will give a brief explanation of the numerical method focusing on a difference from his method. Here the variable $x - \Phi_s/c$ on the free surface

N	a	$\bar{\eta}_s$	λ
$(b, q_c) = (0.03, 0.25505)$			
60	0.0222088233	0.0298327274	0.043868
120	0.0222088210	0.0298327274	0.044600
240	0.0222088209	0.0298327274	0.044604
480	0.0222088209	0.0298327274	0.044605
$(b, q_c) = (0.1, 0.2459)$			
60	0.0671285940	0.0991220137	0.059178
120	0.0671283429	0.0991220218	0.059142
240	0.0671283115	0.0991220225	0.059120
480	0.0671283075	0.0991220226	0.059117

TABLE 1. Convergence of the basic wave (a: crest-to-trough height; $\bar{\eta}_s$: mean surface height) and the growth rates λ of the unstable disturbance mode for $(b, q_c) = (0.03, 0.25505)$ and $(0.1, 0.2459)$. These two cases correspond to the wave amplitudes at $\partial E/\partial c = 0$.

is employed as an independent variable, and unknown variables are represented at discrete mesh points appropriately distributed along the free surface for the range $0 \leqslant x - \Phi_s/c \leqslant 1/2$. By concentrating the mesh points toward a wave crest, one can capture a steep variation of solution near the crest. With a crest at $x - \Phi_s/c = 0$, we employ

$$x - \frac{\Phi_s}{c} = \frac{1}{2}\left(\gamma_i + \frac{P-1}{\pi}\sin \pi\gamma_i\right), \tag{4.1}$$

where $N + 1$ mesh points are distributed at $\gamma_i = i/N$ $(i = 0, 1, 2, \ldots, N)$, and P is a small positive constant. Then $\mathrm{d}\left(x - \Phi_s/c\right)/\mathrm{d}\gamma_i$ is periodic in γ_i and infinitely differentiable with respect to γ_i even at $x - \Phi_s/c = 0$ and $1/2$, where symmetric and periodic conditions are imposed, respectively. The remaining numerical procedure is the same as that explained in Tanaka (1986) so that the reader is referred to that paper. Table 1 shows some numerical results thus obtained: the convergence of the crest-to-trough height $a \equiv \eta_s(0) - \eta_s(1)$ and the mean surface height $\bar{\eta}_s$ of basic waves as well as the growth rates λ of the unstable disturbance mode, as the number N of mesh points increases for given b and q_c defined by

$$q_c \equiv \frac{1}{c}\left[\sqrt{\left(c - \frac{\partial \Phi_s}{\partial x}\right)^2 + \left(\frac{\partial \Phi_s}{\partial y}\right)^2}\right]_{x=0,\, y=\eta_s}, \tag{4.2}$$

are shown. We find that convergence is achieved up to six and seven significant figures for a and $\bar{\eta}_s$ respectively, and up to three significant figures for λ with only $N = 120$.

Figure 1 shows the growth rates λ of the unstable disturbance mode thus obtained for $b = 0.03$ and 0.1. One can clearly see that an exchange of stability occurs not at $\partial E/\partial c = 0$ but at $\partial \bar{E}/\partial c\big|_{\bar{\eta}_s} = 0$. At the wave amplitude corresponding to $\partial E/\partial c = 0$, the basic waves are clearly unstable. The corresponding growth rates λ of the unstable disturbance mode are given in table 1, which shows a good convergence of λ to positive values. Thus, our analytical result that an exchange of stability occurs not at $\partial E/\partial c = 0$ but at $\partial \bar{E}/\partial c\big|_{\bar{\eta}_s} = 0$, has been verified numerically. Table 2 shows

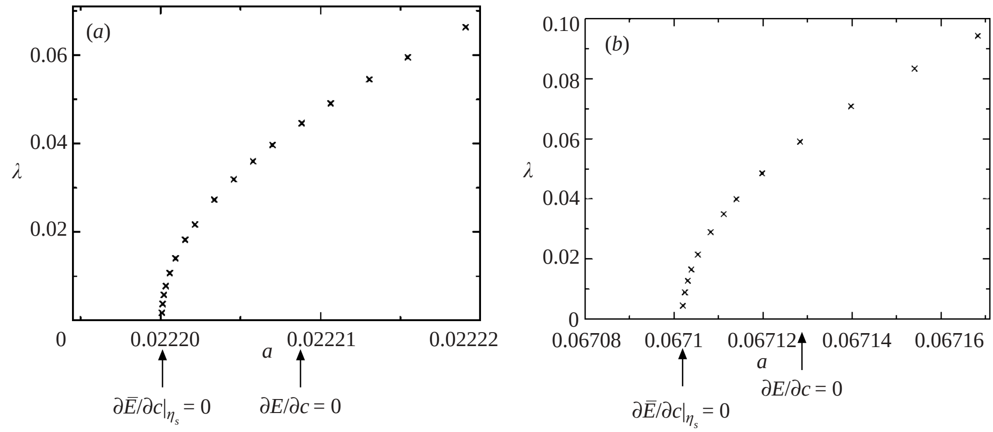

FIGURE 1. Growth rates λ of the unstable disturbance mode versus crest-to-trough height a of the basic wave: (*a*) $b = 0.03$, (*b*) $b = 0.1$. The arrows show the positions of $\partial \bar{E}/\partial c|_{\bar{\eta}_s} = 0$ and $\partial E/\partial c = 0$.

b	a	$\bar{\eta}_s$
0.01	0.0076577	0.0099771
0.02	0.015046	0.019918
0.03	0.022200	0.029833
0.05	0.035913	0.049622
0.1	0.067102	0.099122
0.2	0.11034	0.19912
0.3	0.12837	0.29963
0.5	0.13591	0.49997
∞	0.13660	∞

TABLE 2. Crest-to-trough heights a of surface gravity waves at which an exchange of stability occurs for various values of the Bernouilli constant b. The corresponding mean surface heights $\bar{\eta}_s$ are also shown.

the crest-to-trough heights a of the surface gravity waves at which an exchange of stability occurs for various values of b.

It should be noted that the branch of unstable eigenvalues λ meets the a-axis $\lambda = 0$ at right angles (figure 1). The same feature is seen for periodic waves on deep fluid (Tanaka 1983, 1985). In the case of solitary waves, however, the branch meets the wave-amplitude axis $\lambda = 0$ at acute angles according to numerical studies by Tanaka (1986). An analytical proof is found in Kataoka (2006).

5. Concluding remarks

The linear stability of periodic surface gravity waves on fluid of finite depth to superharmonic disturbances is examined both analytically and numerically. It is found that an exchange of stability occurs at the stationary value of the total wave energy as a function of wave speed for fixed 'mean surface height'. The surface displacement, which constitutes the above total wave energy, must be measured from the 'mean surface height'. It is important to know the law of nature precisely, since it lays the foundations of understanding the physical mechanism.

There is one important unanswered question concerning the mechanism of the superharmonic instability: why does an exchange of stability occur at the extremum in the total wave energy? Longuet-Higgins & Cleaver (1994) showed that the crest of a steep wave, considered in isolation, is linearly unstable, and suggested that the superharmonic instability is caused by the steepness of the crest (Longuet-Higgins, Cleaver & Fox 1994; Longuet-Higgins & Dommermuth 1997). We should seek, however, some physical mechanism that connects the above two fundamentally different characters: the crest instability, which is a local phenomenon, and the extremum in the total wave energy, which is a global quantity.

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